THE INFLUENCE OF THERMAL EFFECTS ON THE VISCOUS RESISTANCE OF A STEADY UNIFORM FLOW OF LIQUID

(VLIIANIE TEPLOVOGO EFFEKTA NA VIAZKOE SOPROTIVLENIE V USTANOVIVSHEMSIA ODNOMERNOM TECHENII KAPEL'NOI ZHIDKOSTI)

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In this note we consider the problem of the dependence between the stress and the motion of the boundary for the case of a steady axial flow of a liquid between two infinite parallel cylindrical surfaces. We take into account the generation of heat due to the dissipation of energy and also the dependence of viscosity of the liquid on temperature.

1. One of the simpler thermohydrodynamic problems is that of the flow between two flat parallel, infinite planes, one of which moves uniformly and with no pressure gradient in the direction of motion.

The elementary solution of this problem is well known. It is not too difficult to obtain somewhat more complicated results, taking into account the heat exchange and the temperature effect on the viscosity of the liquid, but, as before, neglecting the dissipation of the energy. In these cases the energy equation is integrated independently from the equation of motion. In this way the effect of the velocity field on the thermal regime of the flow is neglected, and the viscosity distribution in the liquid layer appears to be independent of the velocity of the boundary. Therefore, it follows from all these solutions that with increasing velocity of the wall U the friction stress increases linearly, $\tau = aU$. It is obvious that accounting for the dependence of the viscosity on dissipative heating of the liquid may substantially change the character of the dependence between τ and U.

In the work by Hagg [1] a solution has been given of the problem of flow between flat parallel planes, taking into account the energy dissipation of the liquid, the viscosity of which depends on temperature according to Reynolds' relation:

$$\eta = \eta_m e^{-\beta(T-T_m)} \qquad (1.1)$$

where β and T_m are constants, and $\eta_m = \eta(T_m)$. The solution for $\tau(U)$ obtained by Hagg for the case of simple boundary conditions indicates that increasing the velocity increases the stress to a certain maximum value, after which the latter gradually decreases, approaching zero asymptotically.

In papers by Pavlin [2] and Targ [3] the same problem has been considered for a different dependence of viscosity on temperature, namely for a hyperbolic law

$$\eta = \eta_m \frac{1}{1+\alpha^2 (T-T_m)} \tag{1.2}$$

It was found by them that for increasing velocity the stress monotonically increases, attaining asymptotically a certain definite value. However, in this case also the analytical solution showed an upper limit on the friction stress for arbitrarily high values of velocity, other conditions remaining constant.

We may note that from the solution of other thermohydrodynamic problems [4,5], where the hyperbolic dependence (1.2) between the friction stress and characteristic velocity of the flow has been also utilized, one can obtain results qualitatively analogous to those of Ref. [2,3].

Golubev [6] was able to show that over a considerable range of velocities, solutions [2,3] agree quite well with experiment. The data of Golubev indicate that the slight change in stress for considerable changes in velocity, shown by the solutions [2,3], may occur under real conditions, approximating to that in a lubricating film.

The demonstration of limitations on the stress for arbitrarily high velocities of a liquid is one of the essential results of thermohydrodynamics. It remains unexplained, however, to what extent this effect is determined by the nature of the assumed dependence of the viscosity on temperature and other assumptions underlying the solution. It is necessary to point out that equations (1.1) and (1.2) assure an exact solution for comparatively small changes of temperature which, of course, corresponds to rather small changes of velocity. Therefore, the use of the formula $\tau = \tau(U)$, from the solutions in Ref. [1,2], for great values of U, (which is essentially equivalent to the extrapolation of equations (1.1) and (1.2) into the region of arbitrarily large temperatures) cannot be considered sufficiently reliable.

In the following we study the dependence between the stress and the velocity of a viscous liquid contained in a narrow gap between two cylindrical surfaces, for an arbitrary dependence of the viscosity on temperature.

2. Consider a layer of a viscous liquid contained between two infinit flat planes $y = 0$ and $y = h$, one of which moves uniformly in direction x with velocity U. As has been shown before, all axial flows in the gap between two arbitrary cylindrical planes can be reduced to such a form [4].

The flow and heat transfer processes which are established in such a system, in the absence of body forces and of pressure gradients in the direction of flow, can be described by the following equations [3].

$$\tau = \eta \frac{dv}{dy}, \qquad \frac{d\tau}{dy} = 0, \qquad \frac{d^2T}{dy^2} + \frac{\tau^2}{Jk\eta} = 0 \tag{2.1}$$

Here $\tau = \tau_{yx}$ is the tangential stress, $v = v_x$ is the flow velocity, k is the thermal conductivity of the liquid, $\eta = \eta_r(T)$ is the viscosity, J is the mechanical equivalent of thermal energy. We shall assume the plane $y = 0$ to be stationary. Then the boundary conditions for the velocity are

$$v(0) = 0, \qquad v(h) = U \tag{2.2}$$

The boundary conditions for the temperature we take to be in the simplest form, assuming the temperatures of both walls are known:

$$T(0) = T_0, \qquad T(h) = T_h \tag{2.3}$$

We introduce into the system (2.1) - (2.3) the dimensionless variable

$$\xi = \frac{y}{h}, \qquad \theta = \frac{T - T_m}{T_m}, \qquad \psi = \frac{\eta_m}{\eta}, \qquad u = \frac{v}{U} \tag{2.4}$$

where T_m is the characteristic temperature, such that $T > T_m$ and $\eta_m = \eta(T_m)$.

We introduce also the dimensionless parameters

$$\Pi_1 = \frac{\tau h}{2Jk\eta_m T_m}, \qquad \Pi_2 = \frac{\tau h U}{JkT_m}, \qquad \Pi_3 = \frac{\Pi_2}{\Pi_1} = U\left(\frac{\eta_m}{2JkT_m}\right)^{1/2} \tag{2.5}$$

The quantity Π_1 represents the dimensionless stress, Π_2 determines the strength of heat generation, Π_3 is a dimensionless characteristic velocity.

In the new variables the system (2.1) has the form:

$$\frac{1}{\psi}\frac{du}{d\xi} = \frac{1}{2}\frac{\Pi_1}{\Pi_3}, \qquad \frac{d^2\theta}{d\xi^2} + \frac{\Pi_1^2}{2}\psi = 0 \tag{2.6}$$

where $\psi = \psi(\theta)$. Eliminating the function ψ from the last equation by means of the first equation (2.6), we obtain

$$\frac{d^2\theta}{d\xi^2} + \Pi_2 \frac{du}{d\xi} = 0 \tag{2.7}$$

The boundary conditions for the velocity u, temperature θ and fluidity ψ, are

$$u(0) = 0, \quad \theta(0) = \theta_0, \quad \psi(0) = \psi_0, \; u(1) = 1, \quad \theta(1) = \theta_1, \; \psi(1) = \psi_1 \tag{2.8}$$

The second equation (2.6) can have its order lowered, and reduces to the following:

$$\left(\frac{d\theta}{d\xi}\right)^2 = \Pi_1^2 \left[F^* - F(\theta)\right] \qquad \left(F(\theta) = \int_0^\theta \psi(\theta) d\theta, \quad F^* = \text{const}\right) \tag{2.9}$$

Equation (2.7) can be also integrated, after which we obtain

$$\frac{d\theta}{d\xi} = \Pi_2 (u^* - u) \qquad (u^* = \text{const}, \tag{2.10}$$

We shall determine the constants u^* and F^* by means of the boundary temperatures. From (2.8) and (2.9) we have

$$\theta_0'^2 = \Pi_1^2 [F^* - F(\theta_0)], \qquad \theta_1'^2 = \Pi_1^2 [F^* - F(\theta_1)] \tag{2.11}$$

Similarly, from (2.10), after squaring both sides of the equation, we find:

$$\theta_0'^2 = \Pi_2^2 u^{*2}, \qquad \theta_1'^2 = \Pi_2^2 = (u^* - 1)^2 \tag{2.12}$$

Solving the system (2.11), (2.12) for u^*, F^*, we obtain

$$u^* = \frac{1}{2}\left[1 + \frac{F(\theta_1) - F(\theta_0)}{\Pi_3^2}\right], \qquad F^* = \frac{\Pi_3^2}{4}\left[1 + \frac{F(\theta_1) - F(\theta_0)}{\Pi_3^2}\right]^2 + F(\theta_0) \tag{2.13}$$

From equations (2.9) and (2.10) it is seen that u^* is the value of the velocity, and F^* is the value of the function $F(\theta)$ at the point $\theta^* = \theta(\xi^*)$, where $d\theta/d\xi = 0$.

From the second equation (2.6) it follows that over the whole range of values of ξ for which the solution has meaning and $\psi(\theta) > 0$, the second derivative of temperature $d^2\theta/d\xi^2$ is strongly negative.

Thus, all extremal temperatures in this domain represent maxima. Therefore, in the interval of interest to us (0,1) there may exist a maximum of temperature, but, of course, only one.

In order for the point ξ^* corresponding to this maximum to be located in that interval (0,1), it is necessary and sufficient that the following inequalities be satisfied.

$$\theta_0' \geqslant 0, \qquad \theta_1' \leqslant 0, \qquad \text{or} \quad 0 \leqslant u^* \leqslant 1 \qquad .$$

From the first of equations (2.13) we obtain finally the condition for the existence of a temperature maximum in the layer of liquid:

$$\Pi_3^2 \geqslant |F(\theta_1) - F(\theta_0)| \tag{2.14}$$

The inequality (2.14) indicates that starting with sufficiently large velocities of the boundary, when Π_3 is large, the maximum temperature always occurs in the interval (0,1). Inasmuch as we are interested in conditions of flow at high velocities, we shall assume in the following that condition (2.14) is satisfied.

We shall now find an integral relationship for determining Π_1. To this end we shall integrate (2.9) with respect to ξ, with the limits $(\theta,\ \xi^*)$ for $\theta^1 > 0$, and the limits $(\xi^*,\ 1)$ for $\theta^1 < 0$:

$$\Pi_1\xi^* = \int_{\theta_0}^{\theta^*} \frac{d\theta}{\sqrt{F^* - F(\theta)}} \quad (\theta' \geqslant 0), \qquad \Pi_1(\xi^* - 1) = \int_{\theta^*}^{\theta_1} \frac{d\theta}{\sqrt{F^* - F(\theta)}} \quad (\theta' \leqslant 0) \tag{2.15}$$

From (2.15) we obtain

$$\Pi_1 = \int_{\theta_0}^{\theta^*} \frac{d\theta}{\sqrt{F^* - F(\theta)}} + \int_{\theta_1}^{\theta^*} \frac{d\theta}{\sqrt{F^* - F(\theta)}} \tag{2.16}$$

or

$$\Pi_1 = J_0 + 2J, \qquad J_0 = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{F^* - F(\theta)}}, \qquad J = \int_{\theta_1}^{\theta^*} \frac{d\theta}{\sqrt{F^* - F(\theta)}} \tag{2.17}$$

where the integral J in (2.17) is obviously non-negative. Equation (2.17) represents the desired relation between parameters Π_1 and Π_3, the latter being related to F^* in accordance with formula (2.13).

3. Before investigating the properties of the integrals entering into equation (2.16), we shall prove that when the inequality (2.14) is fulfilled, the derivative

$$\frac{dF^*}{d\Pi_3} = \frac{\Pi_3}{2}\left\{1 - \frac{[F(\theta_1) - F(\theta_0)]^2}{\Pi_3^4}\right\} \tag{3.1}$$

is non-negative, i.e. $F(\theta^*) = F^*$ is a nondecreasing function of Π_3. In addition, from relation (2.13) it is obvious that $F^* \to \infty$ for $\Pi_3 \to \infty$.

Consequently, inasmuch as we are interested in the relation between Π_1 and Π_3 for great values of Π_3 we may study the behavior of Π_1 as a function of Π_3 for large values of F^*.

In equation (2.17) the integral J_0 has limits independent of F^* and obviously tends to zero for $F^* \to \infty$, independently of the form of the function $F(\theta)$.

We shall present the second integral J (2.17), which is improper, in the form:

$$J = \int_{F(\theta_1)}^{F^*} [F^* - F(\theta)]^{-1/2} \frac{1}{dF/d\theta}\, dF \tag{3.2}$$

Remembering, that $dF/d\theta = \psi(\theta)$, and that for a liquid $(d\eta/dT < 0)$ we always have $\psi \geqslant 1$ for $\theta > 0$ $(T > T_m)$, it can be easily shown that the integral J is convergent.

The problem consists of establishing the relation between the character of the function $F(\theta)$ and the properties of the integral J for $F^* \to \infty$ or $\theta^* \to \infty$.

Introducing the variable $t = \sqrt{F(\theta)/F^*}$, the integral J may be written as follows:

$$J = \int_{t_1}^{1} \frac{\Phi(t\sqrt{F^*})}{\sqrt{1-t^2}}\, dt \qquad \left(\Phi = \Phi(t\sqrt{F^*}) = \frac{\sqrt{F(\theta)}}{\psi}\right) \tag{3.3}$$

Making use of theorems following from the well known lemma of Arzela and of the theorem of the mean value (see for example Ref. 7) it may be shown that:

(a) $\lim J = 0$ for $\theta^* \to \infty$, if $\Phi(\infty) = 0$ and $\Phi(t\sqrt{F^*})$ is limited from above.

(b) $\lim J = \pi\Phi(\infty)$ for $\theta^* \to \infty$, if $\Phi(\infty) < \infty$ and $\Phi(t\sqrt{F^*})$ is limited from above.

(c) $\lim J = \infty$ for $\theta^* \to \infty$ if $\Phi(\infty) = \infty$. (3.4)

Returning to the determination of the function $F(\theta)$, it is seen, that if $\psi(\theta)$ is continuous, which is valid on physical grounds, the function$(\theta)$is also continuous, and in addition for $\theta = 0$ we always have $\Phi = 0$. Therefore, for cases (a) and (b) the boundedness of Φ is assured for arbitrary values of θ and hence of $t\sqrt{F^*}$.

Inasmuch as the functions Φ, corresponding to points (a) (b) and (c) constitute a very considerable class, then with them it is possible to describe all practical possible cases of the dependence of the fluidity ψ on temperature.

In the desired relation between Π_1 and Π_3 according to (2.17) we have:

$$\lim_{\Pi_3 \to \infty} \Pi_1 = \lim_{\theta^* \to \infty} \Pi_1 = 2\pi\Phi(\infty) \tag{3.5}$$

If for $\theta^* \to \infty$, the fluidity ψ tends to a finite value, then the function

$$\Phi(\theta^*) = \frac{1}{\psi(\theta)^*}\left(\int_{\theta_1}^{\theta^*} \psi(\theta)\,d\theta\right)^{1/2]} \tag{3.6}$$

increases without limit. If $\psi(\infty) = \infty$, which occurs in the majority of cases, then eliminating the indeterminacy in the expression for $\Phi(\theta^*)$ we obtain from (3.5) finally

$$\lim_{\Pi_3 \to \infty} \Pi_1 = \pi\sqrt{2} \lim_{\theta^* \to \infty} \left(\frac{d\psi}{d\theta}\right)_{\theta=\theta^*}^{-1/2} \tag{3.7}$$

Thus the behavior of the parameter Π_1 for large values of Π_3 is fully determined by the thermal behavior of the fluidity. From relations (2.17 and (3.7) one can easily obtain the results of Ref. [1-3].

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